JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 87, 460–462 (1982)

On Strong Controllability of Infinite-Dimensional Linear Systems

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Submitted by J. P. LaSalle

If an infinite-dimensional linear system is strongly controllable (i.e., every state can be reached from any state in finite time), then it is strongly controllable in uniform finite time.

Consider the infinite-dimensional linear systems

$$x_n = Ax_{n-1} + Bu_{n-1} \tag{1}$$

and

$$dx/dt = Ax(t) + Bu(t), \qquad x(0) = x_0, \tag{2}$$

with Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ as the state space and the control space, respectively. Either system is called *strongly controllable* if any state x in $\mathscr{X}$ can be reached from any initial state x_0 in finite time (which may depend on x and x_0). If the system is strongly controllable we prove that it is strongly controllable in uniform finite time, i.e., there is an integer n (in the discrete case) or time t (in the continuous case) such that every state x can be reached from any state x_0 in n steps (or in time t). A proof given by Fuhrmann [4] is not valid unless $\mathscr{X}$ and $\mathscr{Y}$ are Hilbert spaces. The proof in [4] is based on an operator theory theorem of Douglas [2] which is not always true if the spaces involved are not assumed to be Hilbert spaces. (In Banach spaces a counterexample due to Douglas is given in Embry [3].)

We start with two lemmas on operators. The notation is standard: the set of all bounded linear operators from a Banach space $\mathscr{X}$ to a Banach space $\mathscr{Y}$ is denoted by $\mathscr{B}(\mathscr{X}, \mathscr{Y})$.

LEMMA 1. *Let $\mathscr{X}$, $\mathscr{Y}_n$ $(n = 1, 2,...)$ be Banach spaces and let $T_n \in \mathscr{B}(\mathscr{Y}_n, \mathscr{X})$. If $\mathscr{X} = \bigcup_n T_n \mathscr{Y}_n$, then $\mathscr{X} = T_m \mathscr{Y}_m$ for some m.*

Proof. Let $\mathscr{S}_n = \{y \in \mathscr{Y}_n : \|y\| < 1\}$ be the open unit ball in $\mathscr{Y}_n$. We have

0022-247X/82/060460–03$02.00/0

$\mathscr{X} = \bigcup_n \bigcup_k A_n(k\mathscr{S}_n)$. By the Baire category theorem, there are positive integers n and k such that $\overline{T_n(k\mathscr{S}_n)}$ contains an open set $\{x \in \mathscr{X}: \|x - x_0\| < r\}$ for some $x_0 \in \mathscr{X}$ and $r > 0$. (The bar indicates closure in the norm topology.) Thus $\overline{T_n(2k\mathscr{S}_n)}$ contains the ball $\mathscr{X}_r = \{x \in X: \|x\| < r\}$. The proof of the open mapping theorem [1, p. 56] shows that $\mathscr{X}_r \subseteq T_n(4k\mathscr{S}_n)$ and so $\mathscr{X} = T_n \mathscr{Y}_n$. ∎

LEMMA 2. *Let* $\mathscr{X}$, $\mathscr{Y}_n$ $(n = 1, 2,\dots)$ *be Banach spaces and let* $T_n \in \mathscr{B}(\mathscr{Y}_n, \mathscr{X})$. *If* $\mathscr{X}$ *is the linear span of the operator ranges* $T_n \mathscr{Y}_n$, *then* $\mathscr{X}$ *is the linear span of finitely many ranges* $\{T_n \mathscr{Y}_n : n = 1, 2,\dots, m\}$ *for some* m.

Proof. For every n, let $\mathscr{Z}_n = \mathscr{Y}_1 \oplus \cdots \oplus \mathscr{Y}_n$ with norm given by $\|(y_1,\dots, y_n)\| = \sum_{j=1}^n \|y_j\|$. Let $R_n: \mathscr{Z}_n \to \mathscr{X}$ be the bounded operator defined by $R_n(y_1,\dots,y_n) = \sum_{j=1}^n T_j y_j$. The assumption that $\mathscr{X}$ is the linear span of $T_n \mathscr{Y}_n$ means that $\mathscr{X} = \bigcup_n R_n \mathscr{Z}_n$. Applying Lemma 1 to R_n establishes the result. (The idea of using direct sums is suggested by a proof of a theorem in Banach algebras due to A. M. Davie (unpublished) shown to me by Peter Rosenthal.) ∎

Let us now consider the discrete system (1), where $A \in \mathscr{B}(\mathscr{X})$ and $B \in \mathscr{B}(\mathscr{Y}, \mathscr{X})$. The solution is given by $x_n = A^n x_0 + \sum_{j=0}^{n-1} A^{n-1-j} B u_j$.

THEOREM 1. *If system* (1) *is strongly controllable, then it is strongly controllable in uniform finite time.*

Proof. The System (1) is strongly controllable if and only if for every x in $\mathscr{X}$, there is a positive integer n and vectors $u_0,\dots, u_n$ in $\mathscr{Y}$ such that $x = \sum_{j=0}^{n-1} A^j B u_j$. This simply means that $\mathscr{X}$ is the linear span of the ranges of $A^j B$ $(j = 0, 1,\dots)$. By Lemma 2, $\mathscr{X}$ must be the linear span of $A^j B$ $(j = 0, 1,\dots, n)$ for some n. This is the desired conclusion. ∎

Next we look at the continuous system

$$dx/dt = Ax(t) + Bu(t), \qquad x(0) = x_0, \tag{2}$$

where $\mathscr{X}$ and $\mathscr{Y}$ are Banach spaces, $B \in \mathscr{B}(\mathscr{Y}, \mathscr{X})$, A is the infinitesimal generator of a strongly continuous semigroup of operators $\{T(t): t \geqslant 0\}$ on $\mathscr{X}$, and u belongs to a function space $\mathscr{F}$ consisting of measurable $\mathscr{Y}$-valued functions on $[0, \infty)$ and satisfying some technical conditions [4]. Instead of listing these conditions we will simply assume that $\mathscr{F}$ is the space of all locally L^1 functions from $[0, \infty)$ into $\mathscr{Y}$. This guarantees the existence of the integrals $\int_0^t T(t-s)\, Bu(s)\, ds$. Furthermore

$$\left\| \int_0^t T(t-s)\, Bu(s)\, ds \right\| \leqslant \sup_{0 \leqslant \zeta \leqslant t} \|T_\zeta\| \, \|B\| \int_0^t \|u(s)\|\, ds.$$

With the appropriate interpretation (see [4]), the solution to (2) is

$$x(t) = T(t)\,x_0 + \int_0^t T(t-s)\,Bu(s)\,ds.$$

THEOREM 2. *If system* (2) *is strongly controllable, then it is controllable in uniform finite time.*

Proof. For every $t > 0$, let S_t be the bounded operator from $L^1([0,t], \mathscr{Y})$ into $\mathscr{X}$ given by

$$S_t u = \int_0^t T(t-s)\,Bu(s)\,ds.$$

Strong controllability is equivalent to the condition that $\mathscr{X} = \bigcup_{t>0} \operatorname{range}(S_t)$. Since $\operatorname{range}(S_t) \subseteq \operatorname{range}(S_n)$ for $t \leqslant n$, we have $\mathscr{X} = \bigcup_{n=1}^{\infty} \operatorname{range}(S_n)$. By Lemma 1, $\mathscr{X} = \operatorname{range}(S_n)$ for some n and so system (2) is strongly controllable in finite time n. ∎

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